§1.5 Solution Sets of Linear Systems

A <u>Homogeneous Linear System</u> is a system of linear equations which can be written as a matrix equation of the form

$$Ax = 0 \qquad \text{(here } 0 \text{ is the zero vector)}$$

These systems are always consistent as $x=0$ is a solution, called the <u>trivial solution</u>. Any nonzero solution is called <u>non-trivial</u>.

<u>Remark</u>

$Ax=0$ has a nontrivial solution if and only if there is at least one free variable.

<u>Example</u>

Describe the solution set for $Ax=0$ where $A=\begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 6 & 4 & 2 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 6 & 4 & 2 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{-6R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -14 & -10 & 0 \\ 0 & -7 & -5 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{14}R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 5/7 & 0 \\ 0 & -7 & -5 & 0 \end{array}\right] \xrightarrow[-3R_2+R_1 \to R_1]{7R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1/7 & 0 \\ 0 & 1 & 5/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced echelon form

not a pivot column

x_3 a free variable!

x_3 is a free variable so there is a nontrivial solution and hence infinitely many nontrivial solutions.

$$\begin{bmatrix} x_1 - \frac{1}{7}x_3 \\ x_2 + \frac{5}{7}x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{7}x_3 \\ -\frac{5}{7}x_3 \\ x_3 \end{bmatrix}$$

Thus the solution set is

$$\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right\} = \left\{x_3\begin{bmatrix} 1/7 \\ -5/7 \\ 1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1/7 \\ -5/7 \\ 1 \end{bmatrix}\right\}$$

Line in $\mathbb{R}^3$ passing through origin.

where x_3 can be <u>any</u> real number

In general, for $Ax=0$

- If free variable(s) then infinitely many nontrivial solutions which can be written as a span
- If no free variables, then $x=0$ the only solution

<u>Example</u>

From MA 261 we know $x_1 + 2x_2 + 3x_3 = 0$ is a plane in $\mathbb{R}^3$. Let's verify this using linear algebra, i.e. describe the solution set to

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \end{array}\right]$$

pivot

x_2 and x_3 free

$$\left\{ x_1 + 2x_2 + 3x_3 = 0 \right. \implies \left\{ x_1 = -2x_2 - 3x_3 \right.$$

Solution set is

$$\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right\} = \left\{\begin{bmatrix} -2x_2-3x_3 \\ x_2 \\ x_3 \end{bmatrix}\right\} = \left\{ x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$$ where x_2, x_3 can be any real numbers

$$= \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$$ which is a plane in $\mathbb{R}^3$.

MA 261 exercise

Verify that the cross product of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, the normal vector of the plane $x_1 + 2x_2 + 3x_3 = 0$.

Why is this the case?

Defn

The expression $x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ above is called a <u>parametric vector equation</u> of the plane.

Defn

<u>Nonhomogeneous linear systems</u> are those of the form $Ax = b$ for $b \neq 0$

<u>Question:</u> How does the solution set of $Ax = b$ relate to the solution set of $Ax = 0$?

Example

Describe the solution set to $Ax = b$ where

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 3 & 6 \\ 4 & 3 & 12 \end{bmatrix} \qquad b = \begin{bmatrix} 17 \\ 19 \\ 23 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 17 \\ 2 & 3 & 6 & 19 \\ 4 & 3 & 12 & 23 \end{array}\right] \xrightarrow[-4R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 17 \\ 0 & -3 & 0 & -15 \\ 0 & -9 & 0 & -45 \end{array}\right] \xrightarrow{-\frac{1}{3} R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 17 \\ 0 & 1 & 0 & 5 \\ 0 & -9 & 0 & -45 \end{array}\right] \xrightarrow[9R_2 + R_3 \to R_3]{-3R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 free!

$$\begin{cases} x_1 + 3x_3 = 2 \\ x_2 = 5 \\ x_3 = \text{free} \end{cases}$$

general solution is

$$X = \begin{bmatrix} 2 - 3x_3 \\ 5 \\ x_3 \end{bmatrix}$$

which we can rewrite as

$$X = \begin{bmatrix} 2 - 3x_3 \\ 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

sometimes use t or s as parameter.

Notice if we let $p = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$$Ap = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 3 & 6 \\ 4 & 3 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+15+0 \\ 4+15+0 \\ 8+15+0 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 23 \end{bmatrix} = b$$

$$Av = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 3 & 6 \\ 4 & 3 & 12 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3+0+3 \\ -6+0+6 \\ -12+0+12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

<u>Theorem</u>

Suppose $Ax = b$ is consistent for some b and let p be any solution. The solution set of $Ax = b$ is the set of all vectors of the form $w = p + v$ where v is a solution of $Ax = 0$.

In our case,

$$\{w\} = \left\{ \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Notice

$$Ax = 0 \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 2 & 3 & 6 & 0 \\ 4 & 3 & 12 & 0 \end{array}\right] \xrightarrow{\text{same row reduction as before}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

has solution set $\text{span}\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Geometrically, this is a parallel shift from the solution set of $Ax=0$ to the solution set of $Ax=b$.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

One can easily show:

- solution set of $Ax=0$ is $\left\{ x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

- $p = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a solution to $Ax=b$

$\Rightarrow$ solution set of $Ax=b$ is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$